18.152 Practice problems for the midterm exam

The midterm exam will take place on March 16th Monday 9:35-10:50.

As an open book exam, during the exam you can see
1. the textbook : Partial Differential Equations in Action by Sandro Salsa,
2. notes, copies, and scratch papers.

However, the following are NOT allowed to use
1. electronic devices including Smartpads
2. the other books except the textbook.

Problem 1. Determine whether the following statements are true or false. *You do not need to verify your answer.*

(A) Given a bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary and a smooth function $g : \overline{\Omega} \to \mathbb{R}$, a smooth harmonic function $u : \overline{\Omega} \to \mathbb{R}$ satisfying $\partial_\nu u = g$ on $\partial\Omega$ is unique, where ν is the outward pointing normal direction to $\partial\Omega$.

Proof. False. Suppose that $u(x)$ is a harmonic function with $\partial_\nu u = g$ on $\partial\Omega$. Then, for any constant c, we have $\Delta(u + c) = \Delta u = 0$ in Ω and $\partial_\nu(u + c) = \partial_\nu u = g$ on $\partial\Omega$. Thus, there are infinitely many solutions. □

(B) Given a smooth function $g : \mathbb{R}^n \to \mathbb{R}$, a smooth solution $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ to the heat equation $\partial_t u = \Delta u$ satisfying $u(x, 0) = g(x)$ is unique.

Proof. False. Recall the Tychonov's counterexample we discussed in class. □

(C) Given a bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary, a smooth superharmonic function $u : \overline{\Omega} \to \mathbb{R}$ (i.e. $\Delta u \le 0$) satisfies

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u.$$

Proof. False. The inequality above holds for subharmonic. For example, if $\Omega = (0, \pi)$ and $u(x) = \sin x$, then $(\sin x)'' = -\sin x \le 0$ in $\Omega = (0, \pi)$, namely superharmonic. However,

$$1 = \max_{[0,\pi]} \sin x > 0 = \max\{\sin 0, \sin \pi\} = \max_{\partial(0,\pi)} \sin x.$$

□

(D) Given a bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary and a smooth negative function $c : \overline{\Omega} \to \mathbb{R}$, a smooth solution $u : \overline{\Omega} \to \mathbb{R}$ to the equation $\Delta u(x) + c(x)u(x) = 0$ satisfies

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} \max\{u, 0\}.$$

Proof. True. See lecture notes Feb26. □

Problem 2. $u : [0, L] \times [0, T] \to \mathbb{R}$ is a smooth solution to the heat equation.
(A) Assume $-u_x(0, t) = u_x(L, t) = 0$, and show

$$\frac{d}{dt} \int_0^L u(x, t)dx = 0.$$

Proof.

$$\frac{d}{dt} \int_0^L u dx = \int_0^L u_t dx = \int_0^L u_{xx} dx = u_x\Big|_0^L = u_x(L, t) - u_x(0, t) = 0.$$

□

(B) Assume $-u_x(0, t) = u_x(L, t) \leq 0$, and show

$$\frac{d}{dt} \int_0^L u(x, t)dx \leq 0.$$

Proof.

$$\frac{d}{dt} \int_0^L u dx = \int_0^L u_t dx = \int_0^L u_{xx} dx = u_x\Big|_0^L = u_x(L, t) - u_x(0, t) \leq 0.$$

□

(C) Assume $u(0, t) = u(L, t) = 0$ and $u(x, 0) \geq 0$, and then show

$$\frac{d}{dt} \int_0^L u(x, t)dx \leq 0.$$

Proof. The maximum principle implies

$$\min_{\overline{Q}_T} u \geq \min_{\partial_p Q_T} u = 0.$$

Thus,

$$u_x(0, t) = \lim_{\delta \to 0} \frac{u(\delta, t) - u(0, t)}{\delta} \geq \lim_{\delta \to 0} \frac{0}{\delta} = 0.$$

In the same manner, $u_x(L, t) \leq 0$. Therefore, the result in (B) yields the desired result. □

Problem 3. Given a bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary and a smooth function $f : \overline{\Omega} \to \mathbb{R}$, a smooth function $u : \overline{\Omega} \to \mathbb{R}$ satisfies $\Delta u = f$ in $\overline{\Omega}$ and $u = 0$ on $\partial\Omega$.

(A) Show that the following inequality holds for any $\epsilon > 0$

$$\int_\Omega |\nabla u|^2 dx \le \frac{1}{\epsilon} \int_\Omega f^2 dx + \frac{\epsilon}{4} \int_\Omega u^2 dx.$$

Proof. By the divergence theorem, we have

$$\int_\Omega u\Delta u + \int_\Omega |\nabla u|^2 = \int_{\partial\Omega} u\nabla u \cdot \nu = 0.$$

On the other hand, AM-GM inequality yields

$$|u\Delta u| \le \frac{\Delta u^2}{\varepsilon} + \varepsilon\frac{u^2}{4} = \frac{f^2}{\varepsilon} + \varepsilon\frac{u^2}{4}.$$

Integrating this inequality over Ω and combining it with the result of the divergence theorem gives

$$\int_\Omega |\nabla u|^2 = -\int_\Omega u\Delta u \le \frac{1}{\varepsilon}\int_\Omega f^2 + \varepsilon\int_\Omega \frac{u^2}{4}.$$

□

(B) Suppose that we have the following Poincaré inequality

$$\int_\Omega |u|^2 dx \le C \int_\Omega |\nabla u|^2 dx,$$

where the constant C only depends on n, Ω. Show that

$$\int_\Omega |\nabla u|^2 dx \le C \int_\Omega f^2 dx,$$

holds for some constant C only depending on n, Ω.

Proof. Suppose the Poincaré inequality that $\int_\Omega u^2 \le C\int_\Omega |\nabla u|^2$. Putting this into the right side of the inequality from part (1) gives

$$\begin{aligned}\int_\Omega |\nabla u|^2 &\le \frac{1}{\varepsilon}\int_\Omega f^2 + \frac{\varepsilon}{4}\int_\Omega u^2 \\ &\le \frac{1}{\varepsilon}\int_\Omega f^2 + \frac{\varepsilon}{4}C\int_\Omega |\nabla u|^2\end{aligned}$$

for all $\varepsilon > 0$. Now if we choose say $\varepsilon = 2/C$ and rearrange, we get

$$\int_\Omega |\nabla u|^2 \le C\int_\Omega f^2.$$

□

Problem 4. Let $g : [0, \frac{\pi}{2}] \to \mathbb{R}$ be a smooth function satisfying

$$g(\theta) = \sum_{n=1}^{\infty} A_n \sin(2n\theta), \qquad g^{(3)}(\theta) = \sum_{n=1}^{\infty} -(2n)^3 A_n \cos(2n\theta),$$

for some constants $\{A_n\}_{n\in\mathbb{N}}$. We define $u(r\cos\theta, r\sin\theta)$ by

$$u(r\cos\theta, r\sin\theta) = \lim_{N\to\infty} S_N(r\cos\theta, r\sin\theta).$$

where

$$\varphi_n(r\cos\theta, r\sin\theta) = \frac{A_n}{r^{2n}} \sin(2n\theta), \qquad S_N(r\cos\theta, r\sin\theta) = \sum_{n=1}^{N} \varphi_n.$$

(A) Show that for each fixed $(r,\theta) \in [1, +\infty) \times [0, \frac{\pi}{2}] = \overline{\Omega}$, $S_N(r\cos\theta, r\sin\theta)$ has the limit $u(r\cos\theta, r\sin\theta)$. In particular, $u(r, 0) = u(0, r) = 0$ and $u(\cos\theta, \sin\theta) = g(\theta)$ on $\partial\Omega$.

Proof. We begin by calculating

$$\int_0^{\frac{\pi}{2}} |g^{(3)}(\theta)|^2 d\theta = \sum_{n=1}^{\infty} \int_0^{\frac{\pi}{2}} 2^6 n^6 A_n^2 |\cos(2n\theta)|^2 d\theta = 16\pi \sum_{n=1}^{N} n^6 A_n^2.$$

We denote $K = \frac{1}{4\sqrt{\pi}}(\int |g^{(3)}|^2)^{\frac{1}{2}} = (\sum_{n=1}^{\infty} n^6 A_n^2)^{\frac{1}{2}}$. In addition, for any $k \geq 2$ we have

$$\sum_{n=M}^{\infty} \frac{1}{n^k} \leq \sum_{n=M}^{\infty} \int_{n-1}^{n} \frac{1}{x^2} dx = \int_{M-1}^{\infty} \frac{1}{x^2} dx = -\frac{1}{x}\bigg|_{M-1}^{\infty} = \frac{1}{M-1}.$$

Let $\varphi_n = A_n r^{-2n} \sin(2n\theta)$ and $S_N = \sum_{n=1}^{N} \varphi_n$. Given $N \geq M$, $r \geq 1$ and $|\sin(2n\theta)| \leq 1$ yield

$$|S_N - S_M| \leq \sum_{n=M}^{N} |A_n| \leq \left(\sum_{n=M}^{N} n^6 A_n^2\right)^{\frac{1}{2}} \left(\sum_{n=M}^{N} \frac{1}{n^6}\right)^{\frac{1}{2}} \leq \frac{K}{\sqrt{M-1}}.$$

Hence, S_N is a Cauchy sequence for each (r,θ). Therefore, S_N converges to u at each point.

Moreover, for $\theta = 0$ or $\theta = \frac{\pi}{2}$, we have $S_N(r, 0) = S_N(0, r) = 0$. Hence, the limits $u(r, 0)$ and $u(0, r)$ are also zero. In addition, $u(\cos\theta, \sin\theta) = g(\theta)$ by definition. □

(B) Show that for each fixed $(r,\theta) \in \overline{\Omega}$, $\frac{\partial}{\partial r} S_N$, $\frac{\partial^2}{\partial r^2} S_N$, $\frac{\partial^2}{\partial \theta^2} S_N$ have the limits $\frac{\partial}{\partial r} u$, $\frac{\partial^2}{\partial r^2} u$, $\frac{\partial^2}{\partial \theta^2} u$, respectively. In particular, $\Delta u = 0$ holds in $\overline{\Omega}$.

Proof. As the proof of (A), we can obtain

$$\left|\tfrac{\partial}{\partial r}S_N - \tfrac{\partial}{\partial r}S_M\right| \le \sum_{n=M}^{N} 2n|A_n| \le 2\left(\sum_{n=M}^{N} n^4 A_n^2\right)^{\frac{1}{2}} \left(\sum_{n=M}^{N} \frac{1}{n^4}\right)^{\frac{1}{2}} \le \frac{2K}{\sqrt{M-1}}.$$

In addition,

$$\left|\tfrac{\partial^2}{\partial r^2}S_N - \tfrac{\partial^2}{\partial r^2}S_M\right| \le \frac{4K}{\sqrt{M-1}}, \qquad \left|\tfrac{\partial^2}{\partial \theta^2}S_N - \tfrac{\partial^2}{\partial \theta^2}S_M\right| \le \frac{4K}{\sqrt{M-1}}$$

Hence, $\frac{\partial}{\partial r}S_N$, $\frac{\partial^2}{\partial r^2}S_N$, $\frac{\partial^2}{\partial \theta^2}S_N$ are Cauchy sequences, and thus have the limits $\frac{\partial}{\partial r}u$, $\frac{\partial^2}{\partial r^2}u$, $\frac{\partial^2}{\partial \theta^2}u$, respectively.

Next, we observe

$$\Delta\varphi_n = \frac{\partial^2}{\partial r^2}\varphi_n + \frac{1}{r}\frac{\partial}{\partial r}\varphi_n + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\varphi_n = 0.$$

Hence, $\Delta S_n = 0$. Therefore, $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$ □

(C) Show that $\lim_{r\to+\infty} r|u(r\cos\theta, r\sin\theta)| = 0$.

Proof.

$$r|u| \le \frac{1}{r}\sum_{n=1}^{\infty} \frac{|A_n|}{r^{2n-2}} \le \frac{1}{r}\sum_{n=1}^{\infty}|A_n| \le \frac{1}{r}\left(\sum_{n=1}^{\infty} n^6 A_n^2\right)^{\frac{1}{2}} \left(1 + \sum_{n=2}^{\infty} \frac{1}{n^6}\right)^{\frac{1}{2}} \le \frac{\sqrt{2}K}{r}.$$

Hence,

$$\limsup_{r\to+\infty} r|u(r\cos\theta, r\sin\theta)| \le \limsup_{r\to+\infty} \frac{\sqrt{2}K}{r} = 0,$$

implies the desired result. □

Problem 5. Ω is a bounded open set in $\mathbb{R}^n$ with smooth boundary and $u(x), f(x)$ are smooth function defined over $\overline{\Omega}$. Suppose that given a constant $p > 0$

$$\int_\Omega \frac{1}{p}|\nabla u|^p(x) + f(x)u(x)dx \le \int_\Omega \frac{1}{p}|\nabla v|^p(x) + f(x)v(x)dx,$$

holds if $v : \overline{\Omega} \to \mathbb{R}$ is a smooth function satisfying $u = v$ on $\partial\Omega$. Then, show that the following equation holds in Ω

$$\operatorname{div}\big(\nabla u\,|\nabla u|^{p-2}\big)(x) = f(x).$$

Proof. Given $x_0 \in \Omega$, we define a function $I : (-\delta, \delta) \to \mathbb{R}$ by

$$I(t) = \int_\Omega \frac{1}{p}\big|\nabla(u(x) + t\eta_\epsilon(x - x_0))\big|^p + f(x)(u(x) + t\eta_\epsilon(x - x_0))dx,$$

where η is a rotationally symmetric positive mollifier with compact support. Then, we choose ϵ small enough so that we have $u(x)+t\eta_\epsilon(x-x_0) = u(x)$ on $\partial\Omega$. Then, by the given condition, we have $I(0) \le I(t)$, and thus $I'(0) = 0$.

Since

$$I'(t) = \int_\Omega \nabla\eta_\epsilon(\nabla u + t\nabla\eta_\epsilon)\big|\nabla u + t\nabla\eta_\epsilon\big|^{p-2} + f\eta_\epsilon dx,$$

we have

$$\begin{aligned} I'(0) &= \int_\Omega \nabla\eta_\epsilon\nabla u|\nabla u|^{p-2} + f\eta_\epsilon dx \\ &= \int_{\partial\Omega} \eta_\epsilon u_\nu|\nabla u|^{p-2}dx + \int_\Omega -\eta_\epsilon \mathrm{div}(\nabla u|\nabla u|^{p-2}) + f\eta_\epsilon dx \\ &= -\int_\Omega \eta_\epsilon\Big[\mathrm{div}(\nabla u|\nabla u|^{p-2}) - f\Big]dx. \end{aligned}$$

Now, we claim that $\nabla u|\nabla u|^{p-2}) - f = 0$ at x_0. If not, without loss of generality we assume $\nabla u|\nabla u|^{p-2}) - f > 0$ at x_0. There exists some small ϵ such that $\mathrm{div}(\nabla u|\nabla u|^{p-2}) - f > 0$ in $B_\epsilon(x_0)$. Then, we have contradiction from

$$0 = I'(0) = -\int_\Omega \eta_\epsilon\Big[\mathrm{div}(\nabla u|\nabla u|^{p-2}) - f\Big]dx < 0.$$

In conclusion, we have $\mathrm{div}(\nabla u|\nabla u|^{p-2}) = f$ in Ω. Since u, f are smooth in $\overline{\Omega}$ we have $\mathrm{div}(\nabla u|\nabla u|^{p-2}) = f$ in $\overline{\Omega}$. □

Problem 6. Suppose that a smooth function $u(x)$ satisfies $\Delta u + nu = 0$ in a bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary.

(A) Show that the maximum principle does not hold for the solution u.

Proof. Consider the function $u(x_1, \cdots, x_n) = \prod_{i=1}^n \sin x_i$ on the domain $\Omega = (0, \pi)^n$. You can check directly that $\frac{\partial^2}{\partial x_i^2}u = -u$ and thus $\Delta u + nu = 0$. On the other hand, u vanishes on the boundary of Ω and it is positive in the interior, so it does not obey the maximum principle. □

(B) Suppose that there exists a positive smooth function w satisfying $\Delta w + nw = 0$ in Ω. Prove that

$$\max_\Omega \tfrac{u}{w} \le \max_{\partial\Omega} \tfrac{u}{w}.$$

Proof. Since w is positive, we can define $v = u/w$. Then,

$$\begin{aligned}\Delta v &= \Delta\left(\frac{u}{w}\right) = \operatorname{div}\left(\frac{\nabla u}{w}\right) - \operatorname{div}\left(\frac{u}{w^2}\nabla w\right) \\ &= \frac{\Delta u}{w} - \frac{2}{w^2}\nabla w \cdot \nabla u - \frac{u}{w^2}\Delta w + 2\frac{u}{w^3}\|\nabla w\|^2.\end{aligned}$$

Applying $\Delta u = -nu$ and $\Delta w = -nw$ yields

$$\begin{aligned}\Delta v &= -\frac{nu}{w} - \frac{2}{w^2}\nabla w \cdot \nabla u + \frac{nu}{w} + 2\frac{u}{w^3}\|\nabla w\|^2 \\ &= -2\nabla\left(\frac{u}{w}\right) \cdot \frac{\nabla w}{w} \\ &= -2\nabla v \cdot \nabla \log w.\end{aligned}$$

Namely,

$$\Delta v + 2\nabla v \cdot \nabla \log w = 0.$$

Thus, by the maximum principle v attains its maximum on the boundary. □

Problem 7. We define a smooth function $\Gamma : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ by

$$\Gamma(x) = \frac{x_1}{\|x\|^2}.$$

(A) Show that Γ is harmonic in $\mathbb{R}^2 \setminus \{0\}$.

Proof. We recall the fundamental solution $\Phi(x) = \log\|x\|$, which satisfies $\Delta\Phi(x) = 0$. Since $\Gamma = -\frac{\partial}{\partial x_1}\Phi(x)$, we have

$$0 = -\frac{\partial}{\partial x_1}\Delta\Phi(x) = \Delta(-\partial_{x_1}\Phi(x)) = \Delta\Gamma(x).$$

We can also calculate directly to obtain

$$\frac{\partial^2}{\partial x_1^2}\Gamma = \frac{\partial}{\partial x_1}\left(\frac{1}{\|x\|^2} - \frac{2x_1^2}{\|x\|^4}\right) = -\frac{2x_1}{\|x\|^4} - \frac{4x_1}{\|x\|^4} + \frac{8x_1^3}{\|x\|^6},$$

and

$$\frac{\partial^2}{\partial x_2^2}\Gamma = \frac{\partial}{\partial x_2}\left(-\frac{2x_1x_2}{\|x\|^4}\right) = -\frac{2x_1}{\|x\|^4} + \frac{8x_1x_2^2}{\|x\|^6}.$$

Hence,

$$\Delta\Gamma = -\frac{8x_1}{\|x\|^4} + \frac{8x_1(x_1^2+x_2^2)}{\|x\|^6} = -\frac{8x_1}{\|x\|^4} + \frac{8x_1}{\|x\|^4} = 0.$$

□

(B) Given a smooth function $f(x)$, the following holds

$$\lim_{r\to 0^+} \frac{1}{r^2}\int_{B_r(0)\setminus\{0\}} f(x)\Gamma(x)dx = \tfrac{1}{2}\pi f_1(0),$$

where $f_1(x) = \frac{\partial}{\partial x_1} f(x)$.

Proof. We define $B_r^0 = B_r(0) \setminus \{0\}$, $B_r^+ = \{(x_1, x_2) \in B_r(0) : x_1 > 0\}$, and $B_r^- = \{(x_1, x_2) \in B_r(0) : x_1 < 0\}$. Then, we have

$$\begin{aligned}\int_{B_r^0} f(x)\Gamma(x)dx &= \int_{B_r^+} f(x)\Gamma(x)dx + \int_{B_r^-} f(x)\Gamma(x)dx \\ &= \int_{B_r^+} f(x_1,x_2)\Gamma(x_1,x_2) + f(-x_1,x_2)\Gamma(-x_1,x_2)dx \\ &= \int_{B_r^+} \left[f(x_1,x_2) - f(-x_1,x_2)\right]\Gamma(x)dx.\end{aligned}$$

By the Taylor's theorem, we have

$$|f(x) - f(0) - x\cdot\nabla f(0)| \le M\|x\|^2$$

for $x \in B_1(0)$ where M is some constant depending on $\sup_{B_1}\|\nabla^2 f\|$. Hence,

$$\begin{aligned}|f(x_1,x_2) - f(0) - x_1 f_1(0) - x_2 f(0)| \le M\|x\|^2, \\ |f(-x_1,x_2) - f(0) + x_1 f_1(0) - x_2 f(0)| \le M\|x\|^2.\end{aligned}$$

Combining them yields

$$|f(x_1,x_2) - f(-x_1,x_2) - 2x_1 f_1(0)| \le 2M\|x\|^2.$$

Hence,

$$\begin{aligned}&\left|\int_{B_r^+} (f(x_1,x_2) - f(-x_1,x_2) - 2x_1 f_1(0))\Gamma(x)dx\right| \\ &\le \int_{B_r^+(0)} 2Mx_1 dx = 2M\int_0^r\int_0^\pi \hat{r}^2(\cos\theta)^2 d\theta d\hat{r} = \tfrac{1}{3}M\pi\hat{r}^3\Big|_0^r = \tfrac{1}{3}M\pi r^3.\end{aligned}$$

On the other hand

$$\begin{aligned}&\int_{B_r^+(0)} 2x_1 f_1(0)\Gamma(x)dx = 2f_1(0)\int_{B_r^+(0)} x_1^2\|x\|^{-2}dx \\ &= 2f_1(0)\int_0^r\int_0^\pi (\cos\theta)^2\hat{r}d\theta d\hat{r} = \tfrac{\pi}{2}f_1(0)r^2.\end{aligned}$$

Therefore,

$$\frac{1}{r^2}\left|\int_{B_r^0} f(x)\Gamma(x)dx - \tfrac{\pi}{2}f_1(0)r^2\right| \le \tfrac{1}{3}M\pi r.$$

Passing r to 0 yields the desired result. □